

CURVATURE-ADAPTED SUBMANIFOLDS OF SYMMETRIC SPACES

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ABSTRACT. Curvature-adapted submanifolds have been extensively studied in complex and quaternionic space forms. This paper extends their study to a wider class of ambient spaces. We generalize Cartan's theorem classifying isoparametric hypersurfaces of spheres to any compact symmetric space. Our second objective is to investigate such hypersurfaces in some specific symmetric spaces. We classify those with constant principal curvatures in the Octonionic planes. Various classification results for hypersurfaces in complex two-plane Grassmannians are also obtained.

1. INTRODUCTION

Let M be a connected hypersurface of a Riemannian manifold \overline{M} , \overline{R} be the Riemannian curvature tensor of \overline{M} and ξ a unit normal vector of M at $p \in M$. The normal Jacobi operator

$$K_\xi := \overline{R}(\xi, \cdot)\xi \in \text{End}(T_p M)$$

of M (with respect to ξ) describes the curvature of the ambient manifold \overline{M} at p , whereas the shape operator A_ξ of M (with respect to ξ) describes the curvature of M as a submanifold of \overline{M} in direction ξ . Both of these are self-adjoint operators, and hence have eigendecompositions. M is said to be curvature adapted if these operators are simultaneously diagonalizable at every point $p \in M$. This means that a common eigenbasis for K_ξ and A_ξ exists at every point, which will generically be denoted by E . This condition can be generalized to submanifolds of higher codimension:

Definition 1.1. A submanifold M of (\overline{M}, g) is said to be curvature-adapted if the following two conditions are satisfied at every point $p \in M$:

- (i) The normal Jacobi operator $\overline{R}(\xi, X)\xi =: K_\xi(X) \in T_p M$ for every unit normal vector field ξ of M and $X \in T_p M$.
- (ii) $A_\xi \circ K_\xi = K_\xi \circ A_\xi$, that is the normal Jacobi and shape operators of M commute.

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For a hypersurface (i) is an immediate consequence of the curvature identities. Gray's Theorem [15] states that any tube around a curvature-adapted submanifold of a locally symmetric space is itself curvature-adapted. Thus understanding how curvature-adapted hypersurfaces of a given symmetric space arise yields information about the condition in general.

In the real space forms it is easy to see that every submanifold is curvature-adapted. For other symmetric spaces the condition is restrictive. Curvature-adapted submanifolds in complex and quaternionic space forms has been a particularly fruitful field of study and there is a substantial body of literature concerned with their classification, of which we just mention [2], [4], [12], [15], [17]. Every complex submanifold of a complex space form satisfies this condition, yielding an important family of examples. In complex space forms the curvature-adapted hypersurfaces coincide precisely with the Hopf hypersurfaces. These are hypersurfaces with the property that $-J\xi$ is an eigenvector of the shape operator, with corresponding principal curvature function denoted α (the Hopf principal curvature). Further canonical examples of curvature-adapted submanifolds in general include hyperspheres in any symmetric space and orbits of Hermann actions.

A second family of submanifolds which have been the focus of much attention are isoparametric hypersurfaces.

Definition 1.2. A smooth function $f : M \rightarrow \mathbb{R}$ is said to *transnormal* if there is a smooth function b such that

$$\|df\|^2 = b(f).$$

A transnormal function is said to *isoparametric* if

$$\Delta f = a(f),$$

for another continuous function a .

The first equation is equivalent to the level sets $f^{-1}(c)$ being parallel, the second to the level sets having constant mean curvatures. The level sets are then said to be isoparametric hypersurfaces. Thus if a hypersurface is isoparametric all parallel hypersurfaces have constant mean curvature. In space forms these arose naturally in the study of geometrical optics and their classification here has a long history. Whilst the question is settled for Euclidean and hyperbolic space, Cartan failed in his attempts to classify them in spheres. Today there is a vast literature on this central problem (see [21] for an excellent survey). Denote by M_t the parallel hypersurfaces at distance t from $M_0 = M$, and the corresponding shape operator at points along the normal geodesic $C_\xi(t)$ passing through a point $p \in M_0$ by $A_\xi(t)$. Observe that if the ambient manifold is symmetric the Riccati equation along $C_\xi(t)$,

$$A'_\xi(t) = (A_\xi(t))^2 + K_\xi(t),$$

simplifies to a family of easy-to-solve ODE's if the hypersurface is curvature-adapted; namely

$$\lambda'_i(t) = \lambda_i^2(t) + \kappa_i^2$$

when \overline{M} is compact. Here, and throughout the paper, λ_i denotes a principal curvature function and κ_i^2 an eigenvalue of the normal Jacobi operator with $\kappa_i \geq 0$. There is an analogous formula if \overline{M} is noncompact. This suggests that it is profitable to study this family of hypersurfaces when one wants to analyze the properties of nearby parallel hypersurfaces.

It is a consequence of this equation that if $M \subset G/K$ is a complete curvature-adapted hypersurface with constant principal curvatures in a rank one symmetric space then it is isoparametric. It is by no means clear if these are the only isoparametric hypersurfaces. Cartan [10] showed that this was indeed the case in the spheres. Wang [22] however showed that in general the theory of isoparametric hypersurfaces and that of hypersurfaces with constant principal curvatures are different in symmetric spaces. Specifically, he discovered families of isoparametric hypersurfaces in $\mathbb{C}P^n$ with nonconstant principal curvatures. He also showed that a Hopf hypersurface $M \subset \mathbb{C}P^n$ is isoparametric if and only if it has constant principal curvatures. The obvious question is whether this is true for the non-compact dual $\mathbb{C}H^n$, or indeed for symmetric spaces generally. This question leads to our first main result:

Theorem 1.3. *Let M be a curvature-adapted hypersurface of a compact symmetric space. Then M is isoparametric if, and only if, it has constant principal curvatures and the eigenvalues of K_ξ corresponding to the basis E are constant on M .*

In rank one symmetric spaces the eigenvalues of K_ξ are constant. Hence for spheres we obtain a new proof of Cartan's theorem. This theorem also yields an intrinsic proof of Wang's theorem in the case of $\mathbb{C}P^n$. Wang's proof used the Hopf fibration $S^{2n+1} \rightarrow \mathbb{C}P^n$ and then applied results from the study of isoparametric functions on spheres.

Our second aim in this paper is to understand the geometry of curvature-adapted hypersurfaces in more general symmetric spaces than have previously been investigated. We investigate the geometry of such hypersurfaces in the octonionic projective and hyperbolic planes, denoted $\mathbb{O}P^2$ and $\mathbb{O}H^2$ respectively, where almost nothing is known. Examples analogous to curvature-adapted hypersurfaces in other rank one symmetric spaces will be uncovered and their principal curvatures calculated. We conjecture there are none beyond these examples. Complete curvature-adapted hypersurfaces with constant principal curvatures are then classified.

Theorem 1.4. *Let $M \subset \mathbb{O}P^2$ denote a complete curvature-adapted hypersurface. Then M has constant principal curvatures if, and only if, M is a principal orbit of a cohomogeneity one action.*

This theorem completes the program initiated in [2], [4] and classifies all complete curvature-adapted hypersurfaces of compact rank one symmetric spaces with constant principal curvatures. Together with the first Theorem, it also classifies the complete curvature-adapted hypersurfaces of $\mathbb{O}P^2$ which are isoparametric.

Finally we study curvature-adapted submanifolds of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. These are rank two Hermitian symmetric spaces with the remarkable property that they are the only closed non-Ricci-flat Riemannian manifolds which admit both complex and quaternionic-Kähler structures, denoted J and \mathcal{J} respectively.

Define $\mathbb{H}\xi(p) := \{J_1\xi(p), J_2\xi(p), J_3\xi(p)\}$, where J_1, J_2, J_3 are a local section of \mathcal{J} at $p \in G_2(\mathbb{C}^{m+2})$. Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2}) = SU(m+2)/S(U(m) \times U(2))$, and again denote nearby parallel hypersurfaces by M_t . There exists an almost Hermitian structure $J_1 \in \mathcal{J}$ and a unit vector $Z \perp \mathbb{H}\xi$ so that at a point $p \in M$

$$J\xi(p) = \cos(\alpha(p))J_1\xi(p) + \sin(\alpha(p))J_1Z(p),$$

where $0 \leq \alpha(p) \leq \frac{\pi}{2}$. Hypersurfaces such that $\alpha(p) \in \{0, \frac{\pi}{2}\}$ are classified in [5]: they are precisely the homogeneous hypersurfaces. Moreover, it can be calculated that they are curvature-adapted. We conjecture there are no more curvature-adapted hypersurfaces in $G_2(\mathbb{C}^{m+2})$. As evidence of this, we show that there generically are no curvature-adapted hypersurfaces satisfying one additional constraint.

Theorem 1.5. *There are no curvature-adapted hypersurfaces of $G_2(\mathbb{C}^{m+2})$ such that $\cos(\alpha(p)) \notin \{0, \frac{3}{5}, \frac{4}{5}, 1\} \forall p \in M$ and either $\langle A_\xi(J_1\xi(t)), J_1\xi(t) \rangle$, or $\langle A_\xi(J_1Z(t)), J_1Z(t) \rangle$, or their ratio is constant along the normal geodesic $C_\xi(t)$ through all points $p \in M$.*

In cases where $\cos(\alpha) \in \{\frac{3}{5}, \frac{4}{5}\}$ the eigenspaces of K_ξ change and have different multiplicity, and our approach fails. However, we hope this result will stimulate further work on the study of curvature-adapted submanifolds in higher rank symmetric spaces.

2. ISOPARAMETRIC HYPERSURFACES

Historically, there have been two definitions of isoparametric hypersurfaces of \overline{M} . Cartan [10], following Somigliana, Levi-Civita and Segre, defined a hypersurface $M \subset \overline{M}$ to be isoparametric if $M \simeq f^{-1}(t)$, where $f : \overline{M} \rightarrow \mathbb{R}$ is an isoparametric function. We will follow this definition. For the alternative, which we define as *weakly isoparametric*, a hypersurface $M \subset \overline{M}$ is required to have all constant mean curvatures for all sufficiently close parallel hypersurfaces M_t , $|t| < \epsilon$. For spheres and complex projective spaces it is known that any weakly isoparametric hypersurface is an open part of the level set of an isoparametric function, so there the definitions are equivalent. We also remark that the exceptional orbits of the Riemannian foliation induced by an isoparametric function on any Riemannian manifold

are minimal submanifolds ([14], [22]): this is one of the few general methods of constructing minimal submanifolds of a Riemannian manifold. Throughout this paper M_ξ refers to the focal set of a hypersurface $M \subset \overline{M}$, and connected components of the focal set are denoted Q_i .

Curvature-adapted hypersurfaces were introduced by d'Atri [12], who observed that all known examples of isoparametric hypersurfaces in rank-one symmetric spaces are curvature-adapted. He also generalized Cartan's fundamental formula for isoparametric hypersurfaces in spheres to any rank one symmetric space. We remark that an elegant proof of his theorem may be deduced from the following two steps: firstly using that M is isoparametric to deduce that the focal manifolds are minimal [23], and secondly calculating the principal curvatures of M in terms of the principal curvatures of Q_1 .

The Riccati equation allows much to be said for curvature-adapted hypersurfaces and the relationships between their geometry and that of nearby parallel hypersurfaces. As such, one expects curvature-adapted hypersurfaces and isoparametric hypersurfaces in symmetric spaces to be intimately related. Our next objective is to justify this statement by establishing Theorem 1.3.

Proof. Let $M^n \subset G/K$ be a complete curvature-adapted hypersurface. If it has constant principal curvatures and the eigenvalues of K_ξ associated to E are constant, it is an easy consequence of the Riccati equation that M is isoparametric.

Conversely, suppose that M is isoparametric, but assume that either: (i) the eigenvalues of K_ξ with respect to E are nonconstant, (ii) the principal curvatures of M are nonconstant, or (iii) both the principal curvatures of M and the eigenvalues of K_ξ , denoted κ_i with respect to E are nonconstant. Only the proof of (iii) will be given; all other cases are analogous. Suppose that the principal curvatures of M are nonconstant. Let $p \in M$ be a fixed point. Assume there is a point $q \in M$ where the principal curvatures differ from p . The strategy of the proof is to consider the Riccati equation at these two points. Since $M =: M_0$ is isoparametric, the sum $\sum_i \lambda_i(0) = c_0$, and similarly for all parallel hypersurfaces M_t , $|t| < \epsilon$, one has $\sum_i \lambda_i(t) = c_t$. Here ϵ is $\sup\{|t| : M_t \text{ is a hypersurface}\}$, and for ease of notation we will assume that $\lambda_i(p)(0) \neq \lambda_j(p)(0)$ for all $i \neq j$. If any principal curvature has multiplicity greater than one, the same proof goes through with some trivial modifications. Therefore $\sum_i \lambda_i(p)(t) = \sum_i \lambda_i(q)(t)$ for all $t, |t| < \epsilon$. But from the Riccati equation one may solve to obtain

$$\lambda_i(p)(t) = \kappa_i \cot(\theta_i(p) - \kappa_i(p)t),$$

where $\kappa_i \cot(\theta_i) = \lambda_i$, and similarly at q . Hence

$$\sum_{i=1}^n \kappa_i(p) \cot(\theta_i(p) - \kappa_i(p)t) - \sum_{i=1}^n \kappa_i(q) \cot(\theta_i(q) - \kappa_i(q)t) = 0$$

for $|t| < \epsilon$. Expanding out the Taylor expansion for \cot around $t = 0$ and grouping coefficients yields a polynomial $F(t) = 0$ for all $|t| < \epsilon$. This cannot vanish unless all coefficients of F vanish. This is equivalent to

$$\sum_{i=1}^n \frac{1}{r_i(p) - t} - \sum_{i=1}^n \frac{1}{r_i(q) - t} - \sum_{j=0}^{\infty} f_j t^j = 0,$$

where $r_i = \frac{\theta_i}{\kappa_i}$ and $f_j \in C^\infty(\mathbb{R})$, for all $t, |t| < \epsilon$.

Let $|r_1(p)| = \min\{|r_i(p)|, |r_i(q)|\}$. Suppose there is a focal set, so $|r_1(p)| = \epsilon$. Multiplying across by $(r_1(p) - t)$ and taking the limit as $t \rightarrow r_1(p)$ yields a contradiction (since $\sum_{j=0}^{\infty} f_j t^j$ is convergent in a compact neighbourhood of $r_1(p)$) unless $r_1(p) = r_1(q)$. Repeating this argument if necessary shows that for all the k principal curvatures which focalize the corresponding term r_i is constant. Thus the isoparametric condition may be rewritten as

$$\frac{1}{r_k(p) - t} = \sum_{i=k+1}^n \frac{1}{r_i(p) - t} - \sum_{i=k}^n \frac{1}{r_i(q) - t} - \sum_{j=0}^{\infty} \tilde{f}_j t^j,$$

where $r_i = \frac{\theta_i}{\kappa_i}$, for all $t, |t| < \epsilon$. Then suppose $|r_k(p)| = \min\{|r_i(p)|, |r_i(q)|, i = k, \dots, n\}$. Since both sides of this equation are infinite power series with constant coefficients, if they agree on an open interval (namely $|t| < \epsilon$) then all coefficients must agree and so they agree on any interval for which both power series converge. This implies that we can repeat the above argument and take the limit as $t \rightarrow r_2(p)$ to again derive a contradiction unless $r_2(p) = r_2(q)$, and so forth.

Repeating this argument yields $r_i(p) = r_i(q)$ for all i . Suppose without loss of generality $\kappa_i(p) > \kappa_i(q)$, so that $\theta_i(p) < \theta_i(q)$. Notice $\theta_i(p)$ and $\theta_i(q)$ have the same sign; we assume without loss of generality they are both negative. Then considering nearby hypersurfaces M_t with $t > 0$ we see from the solution to the Riccati equation that for some $t_0 > 0$ one has $\tilde{\theta}_i(p)(t_0) > 0 > \tilde{\theta}_i(q)(t_0)$, where

$$\tilde{\theta}_i(p)(t_0) = (\theta_i(p) - \kappa_i(p)t_0)$$

and similarly at q . Repeating this calculation with $M = M_{t_0}$ if necessary gives a contradiction, and the proof is complete. \square

Remark. The same proof yields an analogous result in the non-compact case with \coth replacing \cot , under the assumption that $|\lambda_i(p)| \geq |\kappa_i(p)| \forall i$.

Suppose there exists a real number $\epsilon > 0$ such that all hypersurfaces M_t within distance $|t| < \epsilon$ of a given hypersurface M of a symmetric space have constant mean curvatures. Such hypersurfaces are said to be *weakly isoparametric*. Let $\mathbb{K}^n(c)$, $c \neq 0$, $\mathbb{K} = \mathbb{C}$ or \mathbb{H} denote the simply connected manifolds with constant holomorphic (resp. quaternionic) sectional curvature c . For such spaces we obtain a stronger result than above: the classification of weakly isoparametric curvature-adapted hypersurfaces.

Theorem 2.1. *Let $M \subset \mathbb{K}^n(c), c \neq 0$, $\mathbb{K} = \mathbb{C}$ or \mathbb{H} be a curvature-adapted hypersurface. Then M is weakly isoparametric if and only if it has constant principal curvatures.*

The proof is an adaptation of a proof due to Cecil-Ryan [8] for isoparametric hypersurfaces in spheres. We note that for $\mathbb{C}P^n$ this result was already proved by Wang using the Hopf fibration, but again our approach has the advantage of giving an intrinsic proof.

Proof. We just give the proof for $\mathbb{K} = \mathbb{C}$, the case of \mathbb{H} is analogous. Let $M \subset \mathbb{C}^n(c), c \neq 0$ be a Hopf hypersurface. It is obvious from the work of Berndt [3]- Kimura[17] that if M has constant principal curvatures it is weakly isoparametric. Suppose conversely M is weakly isoparametric but does not have constant principal curvatures. Since M is weakly isoparametric, the sum of principal curvatures $\sum_{i=1}^{2n-1} \lambda_i(t)$ is constant on M_t . It is well known that the principal curvature $\lambda_1(t) = \alpha(t)$ corresponding to $-J\xi(t)$ is constant ([16],[19]): subtracting this from the equation yields that $\sum_{i=2}^{2n-1} \lambda_i(t)$ is constant on $M(t)$. Now, observing that all the terms in this equation correspond to an eigenvector of the normal Jacobi operator with eigenvalue ± 1 , we can adopt the proof given in [8]. Differentiating this equation with respect to t yields that $\sum_{i=2}^{2n-1} \lambda_i^2(t)$ is constant on M_t . Differentiating this again forces

$$\frac{-c}{2} \sum_{i=2}^{2n-1} \lambda_i(t) + 2 \sum_{i=2}^{2n-1} \lambda_i^3(t)$$

and hence $\sum_{i=2}^{2n-1} \lambda_i^3(t)$ to be constant on M_t . Iterating this calculation n times, one obtains that $\sum_{i=2}^{2n-1} \lambda_i^k(t)$ is constant on M_t for $k = 1, \dots, n$. But it is known $\alpha = \lambda_1$ is constant. From Newton's identities it follows that $\lambda_i(t)$ are constant on M_t , $i = 1, \dots, 2n - 1$, and we are done. \square

3. THE OCTONIONIC PROJECTIVE AND HYPERBOLIC PLANES

The octonionic (or Cayley) projective plane $\mathbb{O}P^2$ and its noncompact dual $\mathbb{O}H^2$ are intriguing mathematical objects. As rank one symmetric spaces one would expect their geometries to be well-understood, yet they remain mysterious objects. Due to their esoteric nature a brief exposition of their properties is presented. We refer the interested reader to [1] for an in-depth study. \mathbb{O} will denote the octonions, an eight dimensional non-associative division algebra over \mathbb{R} which satisfies the alternative law. \mathbb{O} has a multiplicative identity 1 and a positive definite bilinear form $\langle \cdot, \cdot \rangle$ whose associated norm $\| \cdot \|$ satisfies $\|ab\| = \|a\| \cdot \|b\|$. As is the case for the complex numbers and quaternions, each element $a \in \mathbb{O}$ can be expressed in the form $a = \alpha 1 + a_0$ where $\alpha \in \mathbb{R}$ and $\langle \alpha, a_0 \rangle = 0$. A conjugation map is defined as $a \rightarrow a^* = \alpha 1 - a_0$. This is an anti-automorphism: $(ab)^* = b^* a^*$. To measure nonassociativity there is the associator

$$(a, b, c) := (ab)c - a(bc).$$

Linearization of the associative law $(a, a, c) = (c, a, a) = 0$ yields that $(a^*, b, c) = -(a, b, c)$. This implies that $(a, a^*, b) = -(a^*, a^*, b) = 0$. We define a canonical basis of \mathbb{O} to be any basis of the form $\{1 = J_0, J_1, \dots, J_7\}$ such that

- (i) $\langle J_i, J_j \rangle = \delta_i^j$,
- (ii) $J_i^2 = -1, i \neq 0$,
- (iii) $J_i J_j + J_j J_i = 0, i \neq j, i, j \neq 0$,
- (iv) $J_i J_{i+1} = J_{i+3}$, modulo 7, $i \neq 0$.

The Cayley projective and hyperbolic planes have an *octonionic* structure pointwise. We will outline this construction for $\mathbb{O}P^2$. Firstly $\mathbb{O}P^2 = F_4/Spin(9)$ is a sixteen dimensional rank one symmetric space. Therefore the isotropy representation of $Spin(9)$ acts irreducibly on $T_p \mathbb{O}P^2 = \mathbb{R}^{16}$. This induces a transitive group action on the sphere $S^{15} \subset T_p \mathbb{O}P^2$. But $S^{15} = Spin(9)/Spin(7)$. So $Spin(7)$ fixes a point on S^{15} , or equivalently some vector $X \in T_p \mathbb{O}P^2$. If we restrict the $Spin(9)$ action to $Spin(7) \subset Spin(9)$, then $Spin(7)$ fixes X , and so acts trivially on $\mathbb{R}X \subset T_p \mathbb{O}P^2$. It therefore leaves invariant two subspaces of $\mathbb{R}^{15} \perp \mathbb{R}X$, namely \mathbb{R}^7 and \mathbb{R}^8 . The induced representations of $Spin(7)$ are the standard representation and the spin representation respectively. We now identify $T_p \mathbb{O}P^2$ with $\mathbb{O} \oplus \mathbb{O}$ by equating one copy of the octonions with $\mathbb{R} \oplus \mathbb{R}^7$, and the second copy of \mathbb{O} with \mathbb{R}^8 .

Both $\mathbb{O}P^2$ and $\mathbb{O}H^2$ have unique $Ad(Spin(9))$ -invariant Riemannian metrics up to homothety. The curvature tensors for these manifolds are very different to the other compact rank one symmetric spaces. It was written down for the first time in [7] for $\mathbb{O}P^2$ (resp $\mathbb{O}H^2$) at $T_p \overline{M} = \mathbb{O} \oplus \mathbb{O}$ as

$$\begin{aligned} R((a, b), (c, d))(e, f) = & \frac{\pm 1}{4} (\{4\langle c, e \rangle a - 4\langle a, e \rangle c + (ed)b^* \\ & - (eb)d^* + (ad - cb)f^*\}, \{4\langle d, f \rangle b - 4\langle b, f \rangle d \\ & + a^*(cf) - c^*(af) - e^*(ad - bc)\}). \end{aligned}$$

The inner product \langle, \rangle_p induced by the metric is given by

$$\langle (a, b), (c, d) \rangle = \langle a, c \rangle + \langle b, d \rangle.$$

Throughout, we scale the metric to have sectional curvatures to lie between ± 1 and ± 4 .

Let M be a real hypersurface of $\mathbb{O}P^2$ with normal vector field ξ . Then along the normal geodesic C_ξ we can parallel translate our basis of $T_p M$, $\mathbb{O} \oplus \mathbb{O}$, and hence get an invariant description of the Riemannian curvature tensor along C_ξ in terms of $\mathbb{O} \oplus \mathbb{O}$. This is because the isotropy group of the geodesic C_ξ is $Spin(7) \subset Spin(9)$, and as we have seen the induced action of $Spin(7)$ on $T_p \mathbb{O}P^2 = \mathbb{R}^8 \oplus \mathbb{R}^8$ decomposes into actions on $\{\xi\} \oplus \mathbb{R}^7 \oplus \mathbb{R}^8$. The \mathbb{R}^7 is calculated to correspond to the $+4$ eigenspace of K_ξ and the second copy of \mathbb{R}^8 corresponds to the $+1$ eigenspace of K_ξ . Taking $\xi = (1, 0)$ along C_ξ , we may choose as our common eigenframe along the geodesic C_ξ the

basis

$$E(t) = \{U_i(t), V(t), J_i V(t)\},$$

where J_i is the octonionic structure along C_ξ , $U_i = -J_i \xi$ span the ± 1 eigenspace of K_ξ and $V, J_i V$ span the ± 4 eigenspace of K_ξ . It is to be assumed we are working with this frame in what follows.

Such structures only exist along the geodesic C_ξ : there cannot be a parallel rank seven subbundle $\mathcal{J} \subset \text{End}(T\overline{M})$, $\overline{M} = \mathbb{O}P^2$ or $\mathbb{O}H^2$. This can be seen by the following argument, shown to us by Robert Bryant. If such a bundle were to exist, its holonomy would have to be a quotient group of $\text{Spin}(9)$ that can be embedded in $SO(7)$, but the only such subgroup (since the Lie algebra of $\text{Spin}(9)$ is simple and $\text{Spin}(9)$ is connected) is the trivial subgroup. Thus, if there were such a bundle, it would have a basis of Levi-Civita parallel sections. In particular, the action of $\text{Spin}(9)$ on $T_p \overline{M}$ would have to commute with all of these endomorphisms, meaning that the space of endomorphisms of $T_p \overline{M} = \mathbb{R}^{16}$ which commute with the $\text{Spin}(9)$ action would have dimension at least 7. However the space of linear transformations which commute with $\text{Spin}(9)$ on \mathbb{R}^{16} is one-dimensional, so we derive a contradiction. To see this, suppose that a linear transformation T of \mathbb{R}^{16} commutes with $\text{Spin}(9)$. Thus T acts on \mathbb{R}^{16} commuting with the Lie algebra $\text{spin}(9)$, and so on \mathbb{C}^{16} commuting with the Lie algebra $\text{spin}(9, \mathbb{C}) = \text{spin}(9) + I\text{spin}(9)$, where I denotes the complex structure. Each eigenspace of T must be $\text{Spin}(9, \mathbb{C})$ -invariant. But $\text{Spin}(9, \mathbb{C})$ acts irreducibly on \mathbb{C}^{16} , so T must have the whole of \mathbb{C}^{16} as eigenspace. In other words, T acts as rescaling by a single complex number. T is real, so its eigenvalues come in complex conjugate pairs and so there is a single real eigenvalue. Hence T is a real rescaling.

Proposition 3.1. *The following are curvature-adapted hypersurfaces;*

- (i) *the tube of radius r around a totally geodesic $\mathbb{O}P^k \subset \mathbb{O}P^2$, where $r \leq \frac{\pi}{4}$ and $0 \leq k < 2$,*
- (ii) *the tube of radius r around a totally geodesic $\mathbb{H}P^2 \subset \mathbb{O}P^2$, where $0 < r \leq \frac{\pi}{4}$,*
- (iii) *the tube of radius $r \in \mathbb{R}^+$ around a totally geodesic $\mathbb{O}H^k \subset \mathbb{O}H^2$, where $0 \leq k < 2$,*
- (iv) *the tube of radius $r \in \mathbb{R}^+$ around a totally geodesic $\mathbb{H}H^2 \subset \mathbb{O}H^2$,*
- (v) *a horosphere in $\mathbb{O}H^2$.*

Their principal curvatures, together with their multiplicities, are given in the following table;

M	(i)	(ii)	(iii)	(iv)	(v)
λ_1	-	$\cot(r)$	-	$\coth(r)$	1
λ_2	$-\tan(r)$	$-\tan(r)$	$\tanh(r)$	$\tanh(r)$	-
α_1	$2\cot(2r)$	$2\cot(2r)$	$2\coth(2r)$	$2\coth(2r)$	2
α_2	-	$-2\tan(2r)$	-	$2\tanh(2r)$	-
$m(\lambda_1)$	-	4	-	4	8
$m(\lambda_2)$	8	4	8	4	-
$m(\alpha_1)$	7	3	7	3	7
$m(\alpha_2)$	-	4	-	4	-

Proof. The proof will firstly be outlined in the first two cases: the noncompact cases are exactly analogous. The proof for the horosphere follows the same idea of the proof of Theorem 2 in [3], so we omit it. Fix a point p in $\mathbb{O}P^1$ and a unit normal vector ξ at p . The 4-eigenspace of K_ξ is seven-dimensional and equal to the orthogonal complement of ξ in the normal space of $\mathbb{O}P^1$ at p . The 1-eigenspace is eight-dimensional and is equal to the tangent space of $\mathbb{O}P^1$ at p . This tells us that $\mathbb{O}P^1 \subset \mathbb{O}P^2$ is curvature adapted. It follows that the tubes around $\mathbb{O}P^1$ are curvature-adapted. These tubes are the principal orbits of the action of $Spin(9)$ on $\mathbb{O}P^2 = F_4/Spin(9)$. This is a cohomogeneity one action with two singular orbits; a totally geodesic $\mathbb{O}P^1 = S^8$ and a single point. Alternatively, choosing this point it is curvature-adapted, and so the tubes around this point (geodesic hyperspheres) are curvature-adapted. Hence the tubes around $\mathbb{O}P^1 \subset \mathbb{O}P^2$ are nothing more than geodesic hyperspheres.

For $\mathbb{H}P^2 \subset \mathbb{O}P^2$, note that $\mathbb{H}P^2$ is an orbit of the maximal subgroup $Sp(3)Sp(1)$ in F_4 . However $Sp(1)$ centralizes $Sp(3)$, and so we can restrict to $Sp(3)$. The stabilizer at a point in $\mathbb{H}P^2$ is $Sp(2)Sp(1)$, and the slice representation is the standard representation of $Sp(2)Sp(1)$ on $\mathbb{H}^2 = \mathbb{R}^8$. Fixing a normal vector ξ , the corresponding stabilizer of $Sp(2)Sp(1)$ is $Sp(1)Sp(1)Sp(1)$, and the restriction of the slice representation to this subgroup decomposes into $\mathbb{H} \oplus \mathbb{H}$. We have seen that the eigendecomposition of K_ξ is $\{\xi\} \oplus \{U_i\} \oplus \{J_i V\}$. The intersection with the normal space of $\mathbb{H}P^2$ is

$$\mathbb{H} \oplus \mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^4.$$

This tells us that $K_\xi : T\mathbb{H}P^2 \rightarrow T\mathbb{H}P^2$, and it follows $\mathbb{H}P^2$ is curvature-adapted, as is the tube around it.

We will calculate the principal curvatures for the tube around $\mathbb{H}P^2$. Analogous techniques yields the principal curvatures of all the other examples given. Choose a point q lying at distance r from p along the geodesic C_ξ determined by ξ . Apply again the standard theory to calculate the principal curvatures. Since $\mathbb{O}P^2$ is a symmetric space, the curvature tensor is parallel, so the eigenvalues of K_ξ are constant. From the theory of matrix differential equations this is the same as solving a family of second order ODE's, the first two of which (those involving $X_1, X_2 \in \mathbb{H} \times 1$ tangent to $\mathbb{H}P^2$ are $Y_i'' + 4Y = 0$, with $i = 1, 2$ and initial conditions

$Y_1(0) = 1, Y_1'(0) = 0$. The solution is $\cos(2r)$, and the corresponding eigenvalue $-2\cot(2r)$. Then the second two equations (involving $X_3, X_4 \in \mathbb{H} \times 1$ normal to $\mathbb{H}P^2$ arise from solving $Y_i'' + 4Y = 0$, with $i = 3, 4$ and initial conditions $Y_1(0) = 0, Y_1'(0) = 1$. Solving yields $Y(r) = \sin(2r)$ and the corresponding principal curvature is $-2\tan(2r)$. All principal curvatures are calculated using the same method. \square

We conjecture that these are all the curvature-adapted hypersurfaces in \overline{M} , which is a completely analogous situation to the classification in the quaternionic space forms. Establishing this conjecture would complete the classification of such hypersurfaces in rank one symmetric spaces. To provide some evidence for this conjecture, let us give the proof of Theorem 1.4.

Proof. Let M be a complete curvature-adapted hypersurface of $\mathbb{O}P^2$ or $\mathbb{O}H^2$ with constant principal curvatures. Then it is not hard to see that M must arise as a level set of an isoparametric function. In this situation the focal set has two disjoint connected components

$$M_\xi = Q_1 \cup Q_2.$$

For convenience scale the ambient metric to have ambient sectional curvature lying between 1 and 4 respectively. M may be viewed as a tube around one of its focal manifolds, Q_1 . As M is curvature-adapted, so is M_t for all t . Hence we may choose $U_i \in E(t)$ for all t and by continuity it follows that at the focal manifold $Q_1 = M_{t_0}$ we have $U_i \in E(t_0)$, where $E(t)$ denotes a common eigenbasis of $A_\xi(t) \oplus Id$ and K_ξ . By similar reasoning for every $V \in E(t_0)$, one may choose $J_i V \in E(t_0), i = 1, \dots, 7$ to complete the common eigenbasis that we will work with for this proof. The idea of the proof is to show that either Q_1 or Q_2 is totally geodesic. The tubes around totally geodesic submanifolds which are curvature-adapted are precisely the homogeneous hypersurfaces.

M is equifocal, so a theorem of Tang [20] implies $g = 1, 2$, where $\pi/2g$ denotes the length of the interval between Q_1 and Q_2 . $g = 1$ is immediately ruled out, as any non-zero principal curvature functions on Q_1 would focalize before $t = \frac{\pi}{2}$ unless Q_1 or Q_2 is totally geodesic. Hence it may be assumed that the distance between Q_1 and Q_2 is $\frac{\pi}{4}$. Write $Cot(\theta_i)$ for the nonzero principal curvatures of Q_1 at p with respect to the normal vector $\xi(p)$. The solutions of the Riccati equation fall into two families, for the $+4$ -eigenspace of K_ξ one obtains $\lambda_i(t) = 2\cot(\theta_i - 2t)$ and for the $+1$ -eigenspace $\cot(\theta_i - t)$ with $0 \leq t \leq \frac{\pi}{4}$. Corresponding to $\lambda_i(0) = 0$ principal curvatures at Q_1 the corresponding solutions are $-2\tan(2t)$ and $-\tan(t)$ respectively.

From this it follows all the principal curvatures of Q_1 and Q_2 in the $+4$ -eigenspace must be zero. Moreover, the only possibilities for the principal curvatures of Q_1 in the $+1$ -eigenspace are $1, 0$ or -1 . To see this, observe that each θ_i is either $\frac{\pi}{4}, \frac{\pi}{2}$, or $\frac{3\pi}{4}$ because by assumption the distance between Q_1 and Q_2 is $\frac{\pi}{4}$. This can be seen by travelling along the geodesic C_ξ which

passes through p again at distance π . Suppose Q_1 is not totally geodesic. Then Q_2 is not minimal if it is not totally geodesic, a contradiction. \square

4. COMPLEX TWO-PLANE GRASSMANNIANS

Consider the $(m+1)$ -dimensional $\mathbb{C}P^{m+1}$ embedded canonically as a totally geodesic submanifold of $\mathbb{H}P^{m+1}$. The focal set Q^{m+1} of $\mathbb{C}P^{m+1}$ is a submanifold of codimension three. At each point of Q^{m+1} the null space of the shape operator A_ξ is independent of the choice of normal vector field ξ . It determines a one-dimensional Riemannian foliation \mathcal{F} on Q^{m+1} by closed geodesics (in both the ambient metric of $\mathbb{H}P^{m+1}$ and the induced metric on Q^{m+1}). The orbit space $B^{m+1} := Q^{m+1}/\mathcal{F}$, equipped with the Riemannian structure for which the canonical projection π_B is a Riemannian submersion is isometric to the Riemannian symmetric space $(G_2(\mathbb{C}P^{m+2}), \langle \cdot, \cdot \rangle)$. This fibration yields all the geometric information about $G_2(\mathbb{C}P^{m+2})$ in terms of the intrinsic and extrinsic structure of the focal set Q^{m+1} of $\mathbb{C}P^{m+1}$ in $\mathbb{H}P^{m+1}$. $G_2(\mathbb{C}P^{m+2})$ has both a Kähler structure, J and a quaternionic Kähler structure \mathcal{J} induced from the ambient quaternionic Kähler structure. Let J_1, J_2, J_3 denote a canonical local basis of \mathcal{J} near $p \in M$. Recall that we defined $\mathbb{H}\xi = \text{Span}\{\xi, J_i\xi_p : J_i \in \mathcal{J}_p\}$. Similarly define $\mathbb{C}\xi = \text{Span}\{\xi, J\xi\}$. Then the Riemannian curvature tensor is given as

$$\begin{aligned} R(X, Y, Z) = & \langle Y, Z \rangle X - \langle X, Z \rangle Y \\ & + \langle JY, Z \rangle JX - \langle JX, Z \rangle JZ - 2\langle JX, Y \rangle JZ \\ & + \sum_{\nu=1}^3 \langle J_\nu Y, Z \rangle J_\nu X - \langle J_\nu X, Z \rangle J_\nu Z - 2\langle J_\nu X, Y \rangle J_\nu Z \\ & + \sum_{\nu=1}^3 \langle J_\nu JY, Z \rangle J_\nu JX - \langle J_\nu JX, Z \rangle J_\nu JY \end{aligned}$$

It follows from the expression for the Riemannian curvature tensor that all homogeneous hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are curvature-adapted. These hypersurfaces are

- (i) a principal orbit of the action of $S(U(m+1) \times U(1)) \subset SU(m+2)$,
or
- (ii) a principal orbit of the action of $Sp(n+1) \subset SU(2n+2)$ if $m = 2n$.

In both cases it is possible to calculate that these hypersurfaces are curvature-adapted. For the principal orbits of the action of $S(U(n+1) \times U(1))$, consider the totally geodesic $\mathbb{G}_2(\mathbb{C}^{n+1})$ arising as an exceptional orbit of this action. This is known to be complex with respect to the Kähler structure and quaternionic with respect to the quaternionic-Kähler structure [18]. Hence, a short calculation using the explicit expression for the Riemannian curvature tensor yields that $K_\xi(T\mathbb{G}_2(\mathbb{C}^{n+1})) \subset \mathbb{G}_2(\mathbb{C}^{n+1})$, whence $\mathbb{G}_2(\mathbb{C}^{n+1})$ and the tubes around it are curvature-adapted. The case where M is a principal orbit of the action of $Sp(n+1)$, $n = 2m$ is completely analogous.

Note that both of these families of hypersurfaces may be viewed as tubes of a fixed radius around maximal totally geodesic submanifolds: the second family may be viewed as tubes around a totally geodesic $\mathbb{H}P^n$. Such tubes have at most five distinct principal curvatures, all of which are constant. Therefore the known curvature-adapted hypersurfaces in $G_2(\mathbb{C}^{m+2})$ exhibit similar behaviour as in quaternionic projective and hyperbolic spaces. We now prove Theorem 1.5:

Proof. We will assume without loss of generality that $\langle A_\xi(J_1\xi(t)), J_1\xi(t) \rangle$ is constant along C_ξ : under any of the other assumptions given the proof is analogous. The eigenvalues of K_ξ together with their eigenspaces and dimensions fall into one of three possibilities, all listed in tables in [4]. Suppose we are in the third case of [4], where there exists an almost Hermitian structure $J_1 \in \mathcal{J}$ and a unit vector $Z \perp \mathbb{H}\xi$ so that $J\xi = \cos(\alpha)J_1\xi + \sin(\alpha)J_1Z$, where $0 < \alpha < \frac{\pi}{2}$ and moreover suppose that $\cos(\alpha) \notin \{\frac{3}{5}, \frac{4}{5}\}$. In what follows we will abuse notation slightly by dropping references to the point p . Setting $\beta = \frac{\alpha}{2}$, the eigenspaces corresponding to the eigenvectors $X_1 = \mathbb{R} \cos(\beta)J_1\xi + \sin(\beta)J_1Z$ and $X_2 = \mathbb{R} \sin(\beta)J_1\xi - \cos(\beta)J_1Z$ are seen to both have dimension one. The corresponding eigenvalues are $-4(1 + \cos(\alpha))$ and $-4(1 - \cos(\alpha))$. As there is a common eigenbasis E for A_ξ and K_ξ at each point p , these vectors must both be eigenvectors of A_ξ . Let $A_\xi(X_1) = \lambda_1(X_1)$, etc. Consider the equation

$$(4.1) \quad \langle A_\xi(J_1\xi), J_1Z \rangle = \langle J_1\xi, A_\xi(J_1Z) \rangle.$$

Rewriting the left hand side gives

$$\begin{aligned} \langle A_\xi(J_1\xi), J_1Z \rangle &= \langle A_\xi\left(\frac{-\sin(\beta)}{\cos(\beta)}J_1Z + \frac{1}{\cos(\beta)}X\right), J_1Z \rangle \\ &= \frac{-\sin(\beta)}{\cos(\beta)}\langle A(J_1Z), J_1Z \rangle + \frac{\lambda_1 \sin(\beta)}{\cos(\beta)} \end{aligned}$$

Rewriting the right hand side of the equation produces

$$\begin{aligned} \langle J_1\xi, A_\xi(J_1Z) \rangle &= \langle J_1\xi, A_\xi\left(\frac{1}{\sin(\beta)}X - \frac{\cos(\beta)}{\sin(\beta)}J_1\xi\right) \rangle \\ &= \frac{\lambda_1 \cos(\beta)}{\sin(\beta)} - \frac{\cos(\beta)}{\sin(\beta)}\langle A(J_1\xi), J_1\xi \rangle. \end{aligned}$$

Rearranging for λ_1 yields that

$$\begin{aligned} \lambda_1\left(\frac{\sin(\beta)}{\cos(\beta)} - \frac{\cos(\beta)}{\sin(\beta)}\right) &= \frac{\sin(\beta)}{\cos(\beta)}\langle A_\xi(J_1Z), J_1Z \rangle \\ &\quad - \frac{\cos(\beta)}{\sin(\beta)}\langle A_\xi(J_1\xi), J_1\xi \rangle. \end{aligned}$$

Similarly one can solve Equation (4.1) by the same technique for the eigenvalue $\lambda_2(p)$, but this time rewriting out the left hand side and right hand

side of the equation in terms of X_2 . The reader may check this yields

$$\begin{aligned} -\lambda_2 \left(\frac{\cos(\beta)}{\sin(\beta)} + \frac{\sin(\beta)}{\cos(\beta)} \right) &= \frac{\sin(\beta)}{\cos(\beta)} \langle A_\xi(J_1\xi), J_1\xi \rangle \\ &\quad - \frac{\cos(\beta)}{\sin(\beta)} \langle A_\xi(J_1Z), J_1Z \rangle. \end{aligned}$$

Let M_t denote the hypersurface which is the tube of radius t around M obtained by traveling along the normal geodesic C_ξ . Then α is constant along C_ξ as the ambient space is symmetric by Gray's theorem. Hence along C_ξ we have $J\xi(t) = \cos(\alpha)J_1\xi(t) + \sin(\alpha)J_1Z(t)$. Obviously, if $\xi(t)$ denotes the unit normal vector field we can solve to find $\lambda_1(t)$ and $\lambda_2(t)$. Both these equations holding simultaneously for all t is equivalent to

(4.2)

$$\langle A_\xi(J_1Z(t)), J_1Z(t) \rangle = \lambda_1(t) \left(1 - \frac{\cos^2(\beta)}{\sin^2(\beta)} \right) + \frac{\cos^2(\beta)}{\sin^2(\beta)} \langle A_\xi(J_1\xi(t)), J_1\xi(t) \rangle$$

(4.3)

$$\langle A_\xi(J_1Z(t)), J_1Z(t) \rangle = \lambda_2(t) \left(1 + \frac{\sin^2(\beta)}{\cos^2(\beta)} \right) + \frac{\sin^2(\beta)}{\cos^2(\beta)} \langle A_\xi(J_1\xi(t)), J_1\xi(t) \rangle.$$

Suppose that $\cos(\beta) \neq \sin(\beta)$. Equating $\langle A_\xi(J_1Z(t)), J_1Z(t) \rangle$ in these two equations and performing a routine calculation gives

$$\lambda_1(t) \left(1 - \frac{\cos^2(\beta)}{\sin^2(\beta)} \right) - \lambda_2(t) \left(1 + \frac{\sin^2(\beta)}{\cos^2(\beta)} \right) = (-\cos(2\beta)) \langle A_\xi(J_1\xi(t)), J_1\xi(t) \rangle.$$

As $\langle A_\xi(J_1\xi(t)), J_1\xi(t) \rangle$ is assumed to be constant with respect to t one obtains $\lambda_1(t) = c\lambda_2(t)$ for some constant c .

However, as M_t is a tube around M we can solve for $\lambda_1(t)$ and $\lambda_2(t)$ by solving the Riccati equations explicitly. Choose a point q lying at distance t from p along the geodesic C_ξ . This yields

$$\lambda_1'(t) = (\lambda_1(t))^2 - 4(1 + \cos(\alpha)),$$

$$\lambda_2'(t) = (\lambda_2(t))^2 - 4(1 - \cos(\alpha)).$$

Since $\lambda_1'(t) = c\lambda_2'(t)$, solving the above two equations forces $\lambda_2(t)$ to be constant along C_ξ and hence $\lambda_2'(t) = 0$. But this is impossible, as $0 < \alpha < \frac{\pi}{2}$. If $\cos(\beta) = \sin(\beta)$, the above equations implies $\lambda_2(t) = 0$ for all t , and again the Riccati equation gives a contradiction.

Therefore our supposition must be false, and hence K_ξ must have one of the other two eigendecompositions given in the list in [4]. The reader may read off from these tables that either $J\xi \in \mathcal{J}$ or $J\xi(p) \perp \mathcal{J}(p)$, and hence M has singular normal Jacobi operator at every point. This implies that $\cos(\alpha) \in \{0, \frac{\pi}{2}\}$, a contradiction. \square

For the non-compact dual of $G_2(\mathbb{C}^{m+2})$, namely the symmetric space $SU_{2,m}/S(U_2U_m)$, the same calculation yields

Theorem 4.1. *There are no curvature-adapted hypersurfaces of $SU_{2,m}/S(U_2U_m)$ if $\cos(\alpha(p)) \notin \{0, \frac{3}{5}, \frac{4}{5}, 1\}$ and either $\langle A_\xi(J_1\xi(t)), J_1\xi(t) \rangle$, or $\langle A_\xi(J_1Z(t)), J_1Z(t) \rangle$, or their ratio, is constant along the normal geodesic $C_\xi(t)$ through all points $p \in M$.*

We remark that real hypersurfaces with $\cos(\alpha) \in \{0, \frac{\pi}{2}\}$ are classified in a recent paper [6] up to one possible exception. They obtain the list

- (i) a tube of radius $r \in \mathbb{R}^+$ around a totally geodesic $SU_{2,m-1}/SU_2SU_{m-1}$, or
- (ii) a tube of radius $r \in \mathbb{R}^+$ around a totally geodesic $\mathbb{H}H^n$, where $n = 2m$, or
- (iii) a horosphere whose centre at infinity is singular, or
- (iv) the normal space $\nu(M)$ of M consists of singular tangent vectors X of the form $JX \perp \mathcal{J}X$.

The same calculation as before shows these the first two families of hypersurfaces are curvature-adapted. It is conjectured that there are no hypersurfaces with constant principal curvatures in the fourth case. This is related to a major open problem in the submanifold geometry of symmetric spaces, which is how to find a better understanding of the geometry of horospheres in noncompact symmetric spaces.

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